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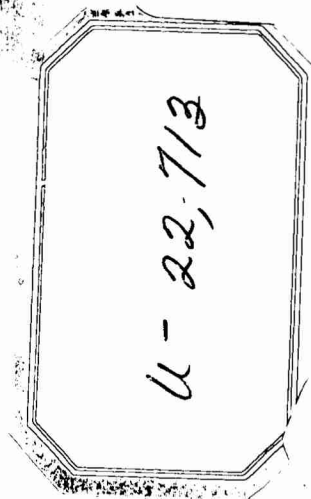
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USING OPERATIONS RESEARCH TECHNIQUES FOR
MINIMUM COST PROCUREMENT

JUNE 1965

Prepared by . . .

OPERATIONS RESEARCH DIVISION (SMUAP-DR)
U.S. ARMY

*Ammunition Procurement and Supply Agency
Joliet, Illinois*

USING OPERATIONS RESEARCH TECHNIQUES FOR
MINIMUM COST PROCUREMENT

BY

Gerald J. Bodoh

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June 1965

Approved by:

A handwritten signature in dark ink, appearing to read "DeWayne W. Lake". The signature is fluid and cursive, with the first name "DeWayne" and last name "Lake" being the most prominent parts.

DeWAYNE W. LAKE

Chief, Operations Research Division
Mgmt Sci and Data Sys Office

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PART I

Building and Using Linear Programming Models

CHAPTER 1

Introduction

In the first chapter we will define Operations Research (OR) and introduce some basic OR techniques and concepts. Among the concepts introduced is that of a scientific model which is the most important tool in an OR analyst's repertoire. This chapter is aimed at serving as a building block for Chapter 2 by giving the reader a feel for Operations Research. It is not intended as a rigorous or theoretical presentation.

Definition of Operations Research

A very brief definition of Operations Research is the utilization of scientific techniques to provide the manager with quantitative information to aid in decision making.

Phases of Operations Research

The general procedure for conducting an OR study is described by the phases which follow:

- a) Formulation of the problem
- b) Construction of a model
- c) Derivation of a solution from the model
- d) Testing the model and solution
- e) Establishing controls over the solution
- f) Implementing the solution

Formulation of the problem is a joint effort by the manager and the OR analyst. First of all, the existence of a problem is recognized by the manager. The manager and the OR analyst then decide upon the objective of the study; the restrictions which must be met (e.g., limited time, limited

resources, etc.); and, a measure of effectiveness of the system to be studied.

Once the problem has been formulated, the OR group attempts to construct a model which adequately expresses the effectiveness of the system in terms of the identifiable variables, both those under the control of management and those not under the control of management.

Viewed generally, a scientific model is a representation of the subject area under investigation (objects, processes, systems, events, etc.) and is used for prediction and control. "It (the model) is intended to make possible, or to facilitate, determination of how changes in one or more aspects of the modelled entity may affect other aspects or the whole."¹ This determination is made by manipulating the model rather than the modelled entity.

There are three types of models available: iconic, analogue and symbolic. An iconic model is characterized by the fact it "looks like" what it represents. This is the kind one encounters most in everyday life. For example, a model airplane or car, a world globe, and a model plant layout are iconic models. An analogue model is characterized by the fact that in it one property is used to represent another. For example a graph is an analogue model which uses distance to represent properties such as time, dollars, weight and quantity. Other examples are maps that use colors to represent geological configurations and slide rules that use distance to represent numbers. Symbolic models use mathematical or logical symbols to represent the components of the modelled entity and their interrelationships. The symbolic model is the most widely used because it lends itself so readily to available techniques

¹Churman, Ackoff & Ackoff (6), p. 157.

of mathematical analysis.

The solution to a model constitutes the set of actions and procedures, relative to the controllable variables, which will result in an optimal overall system effectiveness.² A solution may be found by either analytic or numerical methods. An analytic solution utilizes mathematical methods such as the techniques from algebra and calculus. A numerical solution consists of trying different sets of values in an attempt to find a set of values which yields the best results. The complexity of a numerical solution procedure may vary from a simple trial and error situation to an involved iterative process.

The solution to a model continues to be a feasible solution only as long as the effects and interactions of all variables remain constant and the uncontrollable variables remain within the domain of the model. The significance of a change in the parameters of the model depends upon the resulting change or inaccuracy of the measure of effectiveness. Establishment of controls over the solution consists of the development of a set of rules for a) determining when a significant change occurs and b) modifying the solution if a change occurs.

Prior to implementation both the model and solution should be tested for accuracy and adequacy. Such an evaluation can be accomplished by comparing results obtained when using the model and solution with results obtained without using the model and solution. These determinations can be made by using past data or by a trial run.

²In terms of Game Theory the solution to a model is the selection of that strategy or those strategies which optimize the payoff function.

Implementation requires that the model and solution be transformed into procedures which can be understood and applied by the personnel responsible for their use. This may require changes in existing procedures and resources including the possible necessity for training of personnel. The OR Group makes recommendations regarding implementation. The decision, however, concerning implementation rests with the manager, not the Operations Research Group.

Linear Programming and Nonlinear Programming

Linear Programming and nonlinear programming are techniques for solving certain types of mathematical models. Linear Programming (LP) "... is a technique for allocating or using limited resources (such as plant capacity, storage space and material) to achieve a specific objective (such as least cost, highest profit margin and greatest quantity), where the limited resources have alternate uses and where a change in the amount of resources brings a corresponding or proportional change in result".³ In a linear programming model the interrelationships between the variables which represent the system components are defined by linear equations of the general form $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_{n-1}x_{n-1} + a_nx_n = b$. A linear equation is an equation whose graph is a straight line. It is characterized by the fact that a given change in one variable always produces the same proportional changes in the other variables. Nonlinear programming is a technique for solving mathematical models where at least one of the equations defining the interrelationship between the variables is nonlinear.

³Ferguson and Sargeant (9) pp. 19-20.

CHAPTER 2

Introduction

This chapter may be thought of as being divided into two parts. The first part details the concepts underlying LP, specifies the precise mathematical nature of LP and presents a step-by-step method of model building. The second part illustrates these principles by showing how they apply to a particular example. The example used is a least cost procurement problem for .45 caliber ammunition.

Basic Concepts

Before going into the technique for model building to be presented it might be well to introduce some of the underlying concepts of Mathematical Programming and to detail explicitly the characteristics of a Mathematical Programming model. One of the basic concepts of mathematical programming is the concept of an "activity". An "activity" is a specific method of carrying out a task. Dantzig (7) suggests thinking of activities as "little black boxes" - whose detailed nature we willfully ignore - into which flow inputs and out of which flow outputs. The inputs and outputs of activities are the items of the system. A second basic concept of mathematical programming is the concept of "alternatives". An "alternative" to an activity is another method of carrying out the same task or some part of that task.

The quantity of an activity is called activity level or level of activity. In Dantzig's terms an activity level is the quantity of flow into and out of an activity. Another basic concept is that of a measurable objective

which is to be optimized. Examples of objectives in a scheduling problem might be 1) to minimize cost, 2) to maximize machine usage, 3) to maximize profit, or 4) to minimize idle time.

Two facts follow immediately from the above definitions. First an activity represents a two way classification. In the .45 caliber model an activity will represent a vendor-destination classification - the ammunition purchased from a given vendor for a specific destination. Secondly negative quantities of activities (activity levels) are not possible. For example it is not possible to purchase a negative number of boxes of ammunition.

Constructing a mathematical programming model is a process of stating the relation between alternative activity levels in the form of mathematical expressions (called constraints) and developing a mathematical expression (called an objective function) for the objective to be optimized. The mathematical expressions, usually equations, express algebraically the interdependencies between the various alternate activities and activity levels. An activity level is represented in these mathematical expressions by an x or more generally an x_j .

A linear programming model is a special type of mathematical programming model in which all the constraint relationships and the objective function are linear or capable of being expressed in linear form. There are several methods of converting nonlinear expressions into linear expressions (or of approximating a nonlinear expression with a linear expression) including the "least squares", using linear approximations and

splitting the problem into several parts each of which is very nearly linear.⁴

The Central Mathematical Problem of Linear Programming

The standard form of the linear programming problem is the abstract, symbolic, simulation model which defines the precise nature of the central mathematical problem of any linear programming problem. Any representation of a real system as a mathematical system which exhibits the characteristics prescribed by the formal definition of the standard (form of the) linear programming problem is a linear programming model. However, the terms linear programming problem and linear programming model are often used interchangeably.

A formal definition of the central mathematical problem of linear programming in terms of the basic concepts of LP might be "to determine those non-negative values for the various activity levels that simultaneously satisfy the given linear constraints and optimize the linear objective function".⁵ Restated in symbols this is to

$$\text{Opt} \quad c_1x_1 + c_2x_2 + \dots + c_jx_j + \dots + c_nx_n = Z \quad (1)$$

⁴See Ferguson and Sargeant (9) for a more detailed discussion of this point.

⁵If the restriction of linearity is removed this (word) definition becomes a definition of the general mathematical programming problem; that is, it would then cover the nonlinear case.

Subject to

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1j}x_j + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2j}x_j + \dots + a_{2n}x_n &= b_2 \\
 \vdots & \\
 a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ij}x_j + \dots + a_{in}x_n &= b_i \\
 \vdots & \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mj}x_j + \dots + a_{mn}x_n &= b_m \\
 x_j &\geq 0 \\
 i &= 1, 2, \dots, m \\
 j &= 1, 2, \dots, n
 \end{aligned} \tag{2}$$

or stated in a more general form

$$\text{Opt } \sum_{j=1}^n c_j x_j = Z \tag{1a}$$

$$\text{Subject to } \sum_{j=1}^n a_{ij} x_j = b_i \tag{2a}$$

$$\begin{aligned}
 x_j &\geq 0, \quad i = 1, 2, \dots, m \\
 j &= 1, 2, \dots, n
 \end{aligned}$$

x_j = activity level whose value is to be determined.

c_j = cost of a unit of activity level.

b_i = constant, given as the limit or value of a constraint (equation), sometimes called a stipulation or the stipulated amount.

Equations (1) and (1a) represent different forms of the objective function.

Equations (2) and (2a) represent the constraints (restraints or limits) of the problem.

The authors have taken the liberty of putting the constraints in the definition of the central mathematical problem of LP in the form of equations. Strictly speaking they should have been defined as inequations. However, this would have necessitated a detailed explanation of duality⁶ - which would not add to the reader's understanding of the rest of text. Also, of course, before solving an LP problem the inequations are converted to equations.

The reader should bear in mind that constraints can, indeed very often do, occur in the modelled entity as inequations.

When constraints are inequations of the form "greater than or equal to" (\geq) or "greater than" ($>$) they are generally called restrictions. For example in a procurement problem a vendor may impose a maximum number of pieces he will or can supply. Constraints of the form " \leq " (less than or equal to) or " $<$ " (less than) are called requirements. It is also possible for constraints to be bounded on both sides, that is to restrict the constraint to be less than (or equal to) a stipulated amount but to require that it be greater than (or equal to) a different stipulated amount. We might for example want to purchase at least 1000 boxes

⁶ Although a detailed discussion of duality is inappropriate, a brief, if simplified, explanation is in order. The principle of duality is that related to every linear programming problem, called the direct or primal problem, is another linear programming problem called the dual problem. If in the direct problem the objective function is to be maximized (minimized) then in the dual problem the objective function is to be minimized (maximized). The Dual Theorem, which is the Fundamental Theorem of Linear Programming, states that if either the direct or the dual has a finite optimum solution then both have finite optimum solutions and the values of their optimum solutions are equal. John von Neumann first conjectured this equivalence in October 1947. The theorem was later stated and proved by D. Gale, H. W. Kuhn and A. W. Tucker (10).

of ammunition to meet our needs but no more than 1500 boxes because of limited storage space. When constraints (constraining relationship) occur as inequalities they are converted to equations. The techniques for accomplishing this is discussed in step 11 of the Model Building Method discussed below.

Characteristics of Linear Programming Problems

Linear programming is a quantitative technique for solving management problems. However, not all management problems can be solved by linear programming. Therefore, before considering the problem of building linear programming models it seems worthwhile to first consider the problem of identifying linear programming problems. Fortunately this is usually a relatively simple task. All linear programming problems have the following characteristic:

1. There is a goal to be reached - that is there is a desire to improve present conditions or to achieve the best possible conditions in the future. This goal is represented in the mathematical model by the objective function. Examples are the desire to reduce cost, to increase profit or to increase output.

2. The situation places certain demands upon the solution - that is there are certain conditions of the problem that must be met. In addition the problem imposes limits on certain factors or resources within the problem. The demands are the requirements and the limits are the restrictions of the linear programming problem. They are represented by the constraint equations in the linear programming model.

3. A number of different satisfactory courses of action - paths

to the goal - exist. A satisfactory answer is one that satisfies the requirements within the restrictions.

4. The factors of the problem are quantifiable. Further there are real and measurable differences in the desirability of the various courses of action.

5. At least one of the courses of action must be a best answer. That is, an optimum value of the objective function exists.

A problem must have these characteristics or it cannot be solved by the linear programming technique.⁷ The first and second characteristics follow immediately from the preceding discussion of LP. The third characteristic follows from the fact that if there is only one way of meeting the requirements within the limits then LP is inapplicable because one of the basic assumptions underlying linear programming is that of alternatives. Indeed, if there is only one possible way of fulfilling the requirements the solution to the problem is trivial. The fourth characteristic follows from the fact that linear programming is a technique for obtaining quantitative solutions. The fifth characteristic is self-explanatory.

Introduction to the Model Building Method

In terms of the above discussion we might define linear programming as "... a technique that systematizes for certain conditions the process of selecting the most desirable course of action from a number of available

⁷The characteristics describe any mathematical programming problem. To make the description apply specifically to linear programming the requirement of linearity as per previous discussions must be added.

courses of action, thereby giving management information for making a more effective decision about the resources under its control."⁸

The solutions to LP problems are used by managers as aids in decision making. The LP model is symbolic simulation of a real life system. The validity of the solution depends on how accurately the model describes the system. Determining the solution of the linear programming model is a relatively simple but time consuming and iterative process. This can be simplified by using a standard computer routine to solve the model. However, because the real world is often complex, ambiguous, and sometimes difficult to quantify the process of building a model is not always simple. The process of model building is, in fact, often extremely complex. Indeed, the process of building a model often seems to be more of an art than a science.

Because the validity and applicability of the solution depends directly on how accurately the model simulates the real life system the process of model building is of paramount importance. Yet model building is the aspect of linear programming most neglected in the literature. This is partially due to the fact that it is very difficult to isolate and identify the separate steps that go into the model building process and partially due to the fact that the skilled analyst tends to come to regard model building as almost an intuitive process.

Despite the complexity of the task it is possible to recognize certain principles which distinguish the separate steps in the model building

⁸Ferguson and Sargeant (9) p.3.

process. The procedure outlined below is based on the basic concepts underlying linear programming and the formal definition of the general linear programming problem given above. The procedure is designed to decompose the model building process into separate and distinguishable steps, each involving a single, easily understandable task.

As the reader acquires skill as a model builder he will undoubtedly perform two or more steps simultaneously. Indeed, he will not only combine several into a single step but he will also consider the impact of each step on the final model. It is often possible to simulate part of a system (or a total system) with different and equally valid equations or sets of equations. For example a requirement for an item of GFM (Government Furnish Material) might be expressed as a single equation for the total amount or as several equations - one for each destination. The skilled model builder is able to recognize and evaluate these alternatives.

The reader may feel that the procedure presented is too micro. The authors, however, feel that the micro approach is more than justified by its procedural and conceptual simplicity. We have tried to present a method that is general enough to be applied to any linear programming problem and detailed enough to be readily understandable and useful. The method presented will be especially useful to the novice in the field of LP. However, the more experienced model builder may also profit by gaining insight into model building by being forced to view it as a systematic step by step procedure.

How to Build a Model

The procedure outlined below is designed to incorporate the underlying concepts of linear programming with the formal definition of the general linear programming problem while maintaining ease of understanding. The technique for model building will be demonstrated using the .45 caliber ammunition problem mentioned earlier.

Step 1 - Define the problem in words using the given parameters - requirements, restrictions, cost factors to be considered, etc. This requires a careful study and analysis of the purpose, form, content, and inputs and outputs of the system (situation) to be simulated by the model. The kinds of factors and the number of each should be very explicitly noted. This is the formulation or problem definition phase.

The definition of the problem is given in the paragraph below. Note how the requirements, restrictions, and the factors affecting cost were identified.

.45 Caliber Model - The problem that led to the formulation and solution of the first model was the purchase of .45 caliber ammunition for different destinations. For this problem there are 18 destination points. Each destination point has a given requirement. The total requirement for all destination points can be satisfied by any one of three vendors. The objective is to award contracts in such a way so as to purchase the given requirements for the minimum cost. This objective is influenced by the following factors:

- 1) the unit price of the end item.
- 2) the cost of shipping two different items of Government Furnished Material (GFM) to the vendors (i.e., packing boxes, propellant).

3) the cost of shipping the packaged end item to its destination point(s).

Step 2 - Identify and list the requirements and restrictions which involve the major principle items.⁹ The reader should bear in mind that often separate constraints exist involving the same or identical items. The definition of the problem from Step one should be used as a guide.

The principle item which the .45 caliber ammunition procurement problem is concerned with is .45 caliber ammunition. Referring to the definition from Step 1, you will note that there are eighteen destination points each with its own requirement for ammunition. Therefore there are eighteen requirements for ammunition. Also, there are three vendors and thus three restrictions on capacity. TABLE 1 shows the list of requirements and restrictions for .45 caliber ammunition. A requirement representing the total system requirement for ammunition is also listed.

Step 3 - Analyze the definition of the problem and the list of constraints (from Step 1 and Step 2 respectively) for secondary, subsidiary, constraints.⁹ Subsidiary requirements/restrictions are often implicit and/or dependent on primary requirements or restrictions. For example procurement models for ammunition usually have (implied and dependent)

⁹ Dantzig (7) approaches the problems somewhat differently. He talks in terms of items instead of requirements/restrictions for items. He incorporates Step 2 and Step 3 (above) into a single step which he describes as a process of determining "... the classes of objects, items, which are consumed or produced by the activities ...". The authors, however, feel the approach proposed above is better because it is procedurally and conceptually simpler.

TABLE 1

Requirements and Restrictions for Ammunition

CONSTRAINT	NO.
Destination 1 Requirement	1
Destination 2 Requirement	2
Destination 3 Requirement	3
Destination 4 Requirement	4
Destination 5 Requirement	5
Destination 6 Requirement	6
Destination 7 Requirement	7
Destination 8 Requirement	8
Destination 9 Requirement	9
Destination 10 Requirement	10
Destination 11 Requirement	11
Destination 12 Requirement	12
Destination 13 Requirement	13
Destination 14 Requirement	14
Destination 15 Requirement	15
Destination 16 Requirement	16
Destination 17 Requirement	17
Destination 18 Requirement	18
Vendor 1 Restriction	19
Vendor 2 Restriction	20
Vendor 3 Restriction	21
Total Requirements	22

requirements for GFM. In both Step 2 and Step 3 the list should include possible constraints as well as obvious ones. Later we will winnow out and eliminate unnecessary constraints.

Ask yourself, "Are there any requirements or restrictions that are dependent on those for .45 caliber ammunition?" The definition of the problem shows that there is a requirement for two different GFM for each vendor. Thus we have a total of six requirements for GFM, two each for the three vendors. We also have a total requirement for each GFM giving us a total of eight additional requirements. These are listed in TABLE 2.

TABLE 2

Requirements/Restrictions for GFM

<u>REQUIREMENTS/RESTRICTIONS</u>	<u>NO.</u>
Vendor 1 Requirements for GFM 1	1
GFM 2	2
Vendor 2 Requirements for GFM 1	3
GFM 2	4
Vendor 3 Requirements for GFM 1	5
GFM 2	6
Total Requirements for GFM 1	7
Total Requirements for GFM 2	8

Step 4 - Arrange the data in convenient tabular form. Because of the variety of potential areas of application of LP it is impossible to prescribe an exact format to follow. However, certain general guidelines

can be stated. The stipulated amounts (values) of the requirements and restrictions (the b_i values) should be listed in separate columns or subcolumns and grouped by related item. Put the various "costs" in columns or subcolumns grouped by item.

All columns, subcolumns and entries thereof should be specifically labelled and identified. Make sure you record all the data. At this point it is better to include too much than too little. Extraneous data can be culled out later.

TABLE 3 shows the original data after being arranged according to the guidelines outlined in the instructions. The cost of shipping the end item was added to the cost of the end item for part 1. All the costs in part 1 were rounded to the nearest cent and in part 2 to the nearest one-tenth of a cent. This was done purely as a convenience measure for the purposes of this paper. The original problem was solved using the data to the number of digits given. Rounding did not ^a affect the solution because the differences in prices are large in comparison to the effect of rounding.

Step 5 - The lists of requirements and restrictions from Step 2 and Step 3 contain the possible constraints that the model builder has identified. Review the lists for accuracy and completeness. Taking into consideration the definition of the problem, the limits of the real system and the tabulated data, decide on the necessity and the desirability of each constraint. A necessary constraint is one that imposes a real limit on the model or that has a specific cost attached. The first thing to do is to identify constraints that are not necessary. Unnecessary

TABLE 3

Part 1: Requirements for and Costs of Ammunition - Includes Cost of Shipping End Item

Destination No.	No. Rec'd	Vendor 1	Vendor 2	Vendor 3
1	5,526	46.68	43.03	43.08
2	3,478	47.49	44.29	44.60
3	5,000	47.08	43.75	43.26
4	3,080	46.79	43.49	43.66
5	66	50.10	45.94	47.60
6	150	48.68	45.03	43.89
7	9,043	47.36	44.25	44.40
8	98	48.75	45.09	44.83
9	745	46.99	43.56	42.59
10	14,465	46.99	43.56	42.59
11	14,833	47.36	44.25	44.40
12	1,000	46.93	43.43	43.21
13	1,000	46.79	43.46	43.04
14	1,000	46.94	43.50	42.70
15	1,000	46.86	43.57	43.82
16	1,000	46.88	43.49	43.61
17	1,000	46.48	43.18	43.17
18	1,586	47.29	44.25	44.40

Total Requirement: 64,070 boxes

TABLE 3 (Cont')

Part 2: GFM Requirements and Costs

Vendor	GFM No. 1 \$/Box	GFM No. 2 \$/Lb.	GFM No. 2 \$/Box	Total Cost GFM \$/Box
1	0.131	0.020	0.025	0.156
2	0.087	0.016	0.020	0.107
3	0.099	0.015	0.019	0.118

Total Requirement GFM No. 1: 64,070 Boxes

GFM No. 2: 80,666 Lbs.

constraints should be eliminated unless they are being retained for other purposes.¹⁰ For example, an unnecessary requirement might be retained because it yielded additional information - that is information in addition to the solution - or because it saves work in the evaluation and implementation of the solution. If constraints are being retained to obtain additional information or to save work in implementation of the solution it will be necessary at a later point to evaluate the necessity of adding variables and adjusting coefficients to accomplish this purpose.

Limits imposed by a system can often be represented in a model by different and equally valid equations or sets of equations. Review the lists for alternate methods of simulating part of the system. Evaluate the alternatives. If only one alternative is valid retain it. If all are equally valid retain the one that offers the most in the way of additional benefits, and eliminate the other alternatives. The model builder should be very sure

¹⁰ In Step 10 we will again consider the possibility of modifying the model by eliminating equations. The problem will be approached in Step 10 algebraically whereas in Step 5 the approach is analytical.

that the alternate simulations are equally valid. Once the unnecessary constraints have been eliminated the lists should be reviewed for completeness. That is, the model builder should assure himself that all the limits of the system are explicitly represented by a constraint. A final list of all the requirements and restrictions that are to be included in the model should be prepared.

The requirements for the eighteen destination points listed in TABLE 1 are necessary since they represent actual limits imposed on the model by the real system. On the other hand once these are included the total requirement constraint is not necessary since it does not further delimit the range of possible solutions. The three restrictions on capacity are not necessary because the capacity of each vendor is unlimited so far as we are concerned in this problem.

If one examines the list of requirements for secondary items in TABLE 2 he finds that there are two ways of formating these requirements. The secondary requirements could be included in the model as two constraints each representing the total requirements of all vendors for one GFM or as six constraints each representing the requirement of one vendor for one GFM.

The two methods are equally valid. That is neither is the required one, but at least one must be used. However, the second approach is more convenient. If a variable is added to each of the equations representing the requirements for GFM and coefficients assigned properly the final solution will contain explicitly the amounts of GFM 1 and GFM 2 required for

each vendor. This has the advantage of saving hand computations. It also has the advantage of serving as a reminder that these are indeed individual requirements. Consideration should be given to include both formulations. Since using the two total requirements offers us no additional information nothing is to be gained by including them. Therefore we eliminate them. TABLE 4 shows the revised list of principle and secondary requirements.

Step 6 - List and identify the activities of the system. An activity should be included in this list if and only if it is a possible activity. For example, if a certain vendor will not or cannot supply a given destination then this activity is impossible. The decisions made in Step 5 must be kept in mind. If constraints were deleted or one has decided to obtain extra information by adding constraints or variables the list of variables must be adjusted accordingly. For most problems it is easiest and best to summarize activities in the activity table.

The possible activities for the .45 caliber procurement problem are listed and labelled in columns 1 of TABLE 5. Note how the principle of summarizing was used. For example, the purchases for all possible destination points from one vendor are shown on a single line. The same data, not summarized, is shown in TABLE 5A. Note how much longer this latter table is even using the mathematical shorthand convention of the ellipsis. However, the longer table has the advantage of making it easier to identify the variables that represent the alternate activities when we write the mathematical expressions to represent the interrelationship between the variables in Step 8.

TABLE 4

Requirements for Ammunition and GFM

Req No.	Description
1	Destination 1 Requirement for Ammunition
2	" 2 " " "
3	" 3 " " "
4	" 4 " " "
5	" 5 " " "
6	" 6 " " "
7	" 7 " " "
8	" 8 " " "
9	" 9 " " "
10	" 10 " " "
11	" 11 " " "
12	" 12 " " "
13	" 13 " " "
14	" 14 " " "
15	" 15 " " "
16	" 16 " " "
17	" 17 " " "
18	" 18 " " "
19	Vendor 1 Requirement for GFM 1
20	Vendor 2 Requirement for GFM 1
21	Vendor 3 Requirement for GFM 1
22	Vendor 1 Requirement for GFM 2
23	Vendor 2 Requirement for GFM 2
24	Vendor 3 Requirement for GFM 2

TABLE 5

<u>Explanation (Activity)</u>	<u>Variables</u>
Ammunition from	
Vendor 1 to Destination 1 - 18	$x_1 - x_{18}$
Vendor 2 to Destination 1 - 18	$x_{19} - x_{36}$
Vendor 3 to Destination 1 - 18	$x_{37} - x_{54}$
GFM	
GFM 1 for Vendor 1	x_{55}
GFM 1 for Vendor 2	x_{56}
GFM 1 for Vendor 3	x_{57}
GFM 2 for Vendor 1	x_{58}
GFM 2 for Vendor 2	x_{59}
GFM 2 for Vendor 3	x_{60}

TABLE 5A

<u>Activities</u>	<u>Variables</u>
Vendor 1 to	
Destination 1	x_1
Destination 2	x_2
Destination 3	x_3
.	.
.	.
.	.
Destination 17	x_{17}

TABLE 5A (Cont')

Activities (Cont')	Variables (Cont')
Destination 18	x_{18}
Vendor 2 to	
Destination 1	x_{19}
Destination 2	x_{20}
Destination 3	x_{21}
.	.
.	.
.	.
Destination 17	x_{35}
Destination 18	x_{36}
Vendor 3 to	
Destination 1	x_{37}
Destination 2	x_{38}
Destination 3	x_{39}
.	.
.	.
.	.
Destination 17	x_{53}
Destination 18	x_{54}
GFM 1 to	
Vendor 1	x_{55}
Vendor 2	x_{56}
Vendor 3	x_{57}

TABLE 5A (Cont')

<u>Activities (Cont')</u>	<u>Variables (Cont')</u>
GFM 2 to	
Vendor 1	x_{58}
Vendor 2	x_{59}
Vendor 3	x_{60}

Step 7 - Assign variables to the activities listed to represent the unknown (and to-be-determined) activity levels. No variable should be assigned to an impossible activity. However, if an impossible activity has been listed and is assigned a variable it will take on a zero coefficient in Step 11.

Column labelled "Variables" of TABLE 5 shows the assignment of variables to possible activities.

Step 8 - Develop a mathematical expression for each constraint that shows the relationship of the stipulated amount of that constraint to the sum of the terms formed by multiplying every activity level (representing alternate activities) in that expression by the associated coefficient. That is, write down mathematical expressions to express the interdependencies between the various X_j . The table developed in Step 6 will be an aid in identifying the alternate activities. The coefficients and stipulated amounts will be as yet undetermined and appear in the form of a_{ij} and b_i respectively. Write an equation for the objective function in the form specified in the definition of the general LP problem above. Any variable

that appears in any constraint will appear in the objective function with the appropriate "cost" coefficient. The C_j , cost values, will also be unknown at this point in time.

The mathematical expression for the constraints and the objective function are shown in Figure 1. To illustrate how these mathematical expressions are developed let us examine the first equation in Figure 1 which is $a_{1'1}x_1 + a_{1'19}x_{19} + a_{1'37}x_{37} = b_1$. This says that the sum of the number of boxes of ammunitions bought from the three vendors for destination one must equal the required amount of ammunition for that destination. A similar analysis applies to the other constraints defining requirements for ammunition.

Equations 19 - 24 are concerned with the requirements for GFM. Each equation represents the total requirement of a given vendor for one type of GFM. They state that the total amount of a particular GFM required by the given vendor must equal the sum of the individual requirements (one for each destination) of that vendor for that GFM. The extra term in each GFM equation is the additional variable mentioned in Step 5 that was added to enable us to use these equations to obtain the amount of each GFM required by each vendor directly from the solution.

The objective function states that the sum of the number of boxes purchased from a vendor for a destination times the cost per box, for all possible vendor - destination combination, together with the amount of GFM for each vendor times its cost per unit equals the total cost.

FIGURE 1

Constraints

- (1) $a_{1,1}x_1 + a_{1,19}x_{19} + a_{1,37}x_{37} = b_1$
- (2) $a_{2,2}x_2 + a_{2,20}x_{20} + a_{2,38}x_{38} = b_2$
- (3) $a_{3,3}x_3 + a_{3,21}x_{21} + a_{3,39}x_{39} = b_3$
- (4) $a_{4,4}x_4 + a_{4,22}x_{22} + a_{4,40}x_{40} = b_4$
- (5) $a_{5,5}x_5 + a_{5,23}x_{23} + a_{5,41}x_{41} = b_5$
- (6) $a_{6,6}x_6 + a_{6,24}x_{24} + a_{6,42}x_{42} = b_6$
- (7) $a_{7,7}x_7 + a_{7,25}x_{25} + a_{7,43}x_{43} = b_7$
- (8) $a_{8,8}x_8 + a_{8,26}x_{26} + a_{8,44}x_{44} = b_8$
- (9) $a_{9,9}x_9 + a_{9,27}x_{27} + a_{9,45}x_{45} = b_9$
- (10) $a_{10,10}x_{10} + a_{10,28}x_{28} + a_{10,46}x_{46} = b_{10}$
- (11) $a_{11,11}x_{11} + a_{11,29}x_{29} + a_{11,47}x_{47} = b_{11}$
- (12) $a_{12,12}x_{12} + a_{12,30}x_{30} + a_{12,48}x_{48} = b_{12}$
- (13) $a_{13,13}x_{13} + a_{13,31}x_{31} + a_{13,49}x_{49} = b_{13}$
- (14) $a_{14,14}x_{14} + a_{14,32}x_{32} + a_{14,50}x_{50} = b_{14}$
- (15) $a_{15,15}x_{15} + a_{15,33}x_{33} + a_{15,51}x_{51} = b_{15}$
- (16) $a_{16,16}x_{16} + a_{16,34}x_{34} + a_{16,52}x_{52} = b_{16}$
- (17) $a_{17,17}x_{17} + a_{17,35}x_{35} + a_{17,53}x_{53} = b_{17}$
- (18) $a_{18,18}x_{18} + a_{18,36}x_{36} + a_{18,54}x_{54} = b_{18}$
- (19) $a_{19,1}x_1 + a_{19,2}x_2 + a_{19,3}x_3 + \dots + a_{19,18}x_{18} + a_{19,55}x_{55} = b_{19}$
- (20) $a_{20,19}x_{19} + a_{20,20}x_{20} + \dots + a_{20,36}x_{36} + a_{20,56}x_{56} = b_{20}$
- (21) $a_{21,37}x_{37} + a_{21,38}x_{38} + \dots + a_{21,54}x_{54} + a_{21,57}x_{57} = b_{21}$

FIGURE 1 (Cont')

$$(22) \ a_{22,1}x_1 + a_{22,2}x_2 + \dots + a_{22,18}x_{18} + a_{22,58}x_{58} = b_{22}$$

$$(23) \ a_{23,19}x_{19} + a_{23,20}x_{20} + \dots + a_{23,36}x_{36} + a_{23,59}x_{59} = b_{23}$$

$$(24) \ a_{24,37}x_{37} + a_{24,38}x_{38} + \dots + a_{24,54}x_{54} + a_{24,60}x_{60} = b_{24}$$

Objective Function

$$(25) \ C_1x_1 + C_2x_2 + C_3x_3 + \dots + C_jx_j + \dots + C_{59}x_{59} + C_{60}x_{60} = Z$$

Step 9 - Determine the coefficients of the variables and value (stipulated amount) of the stipulation for each equation. The variables are all in one unit of measure. All of the terms (the products of the activity levels by coefficients) for a given constraint and the stipulated amount of that constraint must be in the same unit of measure. To establish the same unit of measure within a constraint calculate conversion factors and then multiply either the stipulated amount or the terms by these factors. For example, the requirements (the b_i values) might be given in numbers of pieces, the restrictions (b_i values) in man/machine hours, and the variables in pieces. To establish the same unit of measure within a restriction we could calculate the hours per piece for each item in that equation and multiply each of its terms by the appropriate factor. This would convert all the units of measure in that restriction to hours. On the other hand we could calculate the pieces per hour for the item and multiply the stipulated amount given in hours by this factor. This would result in the using pieces for the unit of measure. The latter approach would not work if we had different items with different production rates in the same constraint.

The earlier decision (Step 5) to use or not to use the model to obtain extra information affects the determination of values for coefficients and stipulations. If we have decided or decide at this time to do this, we must add extra variables and adjust the values of the coefficients and/or values of the stipulation accordingly. Once the numerical values of the coefficients and stipulations have been found, rewrite the constraints replacing each a_{ij} and b_i with the proper numerical value.

The constraints with numerical values substituted for the coefficients and stipulated amounts are given in Figure 2. The stipulated amounts for the first eighteen equations are taken directly from TABLE 3. These are the requirements for the eighteen destination points. To see how the coefficients for these equations were determined let us look at the first equation in Figure 1. This equation says that if we multiply the number boxes purchased from vendor 1 for destination 1 by a constant, add the product of the number of boxes purchased from vendor 2 by a second constant and add to this sum the product formed by multiplying the number of boxes purchased from vendor 3 by another constant the sum equals the stipulated amount, which is 5,526 boxes. For each box purchased from a vendor for a destination one box is shipped to that destination. Therefore each of the three constants must be a positive one. A similar analysis shows that all the coefficients of the first eighteen equations must be positive ones. To illustrate a situation where some of the coefficients might not be positive ones let us assume that equation 1 represents a requirement for a different kind of ammunition that is packed 2,000 to box. If the stipulated amount was given in boxes of 1,000 rounds we could either divide the

stipulated amount by two or use two as coefficients for all the terms in equation 1.

The coefficients and stipulated amounts of the three requirements for GFM boxes present a situation that is somewhat different. The values of these stipulations are not known, only the total for the three vendors is known. The values of these stipulations, which will be the requirements of the vendors for GFM boxes, can be obtained from the solution to the model. To show how this is done let us look at equation 19 in Figure 2. The initial value of b_{19} is set to zero, the coefficients of x_1 thru x_{18} set to negative ones and the coefficient of x_{55} to positive ones. This forces x_{55} to assume a value equal to the sum of the values of x_1 thru x_{18} . Thus x_{55} in the solution will take a value equal to the requirement of vendor 1 for GFM 1. A similar analysis applies to equation 20 and equation 21. The requirements for the propellant poses a different problem. For equations 22-24 in Figure 1, a similar analysis to that used for equations 19-21 seems to apply. However, an adjustment in units or in unit cost must be made because costs and total requirement was given in terms of pounds whereas the variables are in terms of numbers of 1,000 round boxes. We can either convert the cost to dollars per box or convert all the terms in these three requirements to pounds. If the former approach is used then the values of x_{58} - x_{60} in the solution will be in terms of boxes and would have to be multiplied by pound per box to convert these figures to pounds in order to implement the solution. The second approach involves using decimal coefficients for all the terms in equations 58-60. For convenience sake we decided to use the first

Figure 2

- (1) $x_1 + x_{19} + x_{37} = 5,526$
- (2) $x_2 + x_{20} + x_{38} = 3,478$
- (3) $x_3 + x_{21} + x_{39} = 5,000$
- (4) $x_4 + x_{22} + x_{40} = 3,080$
- (5) $x_5 + x_{23} + x_{41} = 66$
- (6) $x_6 + x_{24} + x_{42} = 150$
- (7) $x_7 + x_{25} + x_{43} = 9,043$
- (8) $x_8 + x_{26} + x_{44} = 98$
- (9) $x_9 + x_{27} + x_{45} = 745$
- (10) $x_{10} + x_{28} + x_{46} = 14,465$
- (11) $x_{11} + x_{29} + x_{47} = 14,833$
- (12) $x_{12} + x_{30} + x_{48} = 1,000$
- (13) $x_{13} + x_{31} + x_{49} = 1,000$
- (14) $x_{14} + x_{32} + x_{50} = 1,000$
- (15) $x_{15} + x_{33} + x_{51} = 1,000$
- (16) $x_{16} + x_{34} + x_{52} = 1,000$
- (17) $x_{17} + x_{35} + x_{53} = 1,000$
- (18) $x_{18} + x_{36} + x_{54} = 1,586$
- (19) $-x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - x_{14} - x_{15} - x_{16} - x_{17} - x_{18} + x_{55} = 0$
- (20) $-x_{19} - x_{20} - x_{21} - x_{22} - x_{23} - x_{24} - x_{25} - x_{26} - x_{27} - x_{28} - x_{29} - x_{30} - x_{31} - x_{32} - x_{33} - x_{34} - x_{35} - x_{36} + x_{56} = 0$
- (21) $-x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - x_{43} - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - x_{49} - x_{50} - x_{51} - x_{52} - x_{53} - x_{54} + x_{57} = 0$

Figure 2 (Cont')

$$\begin{aligned}
 (22) \quad & -x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - \\
 & x_{14} - x_{15} - x_{16} - x_{17} - x_{18} + x_{58} = 0 \\
 (23) \quad & -x_{19} - x_{20} - x_{21} - x_{22} - x_{23} - x_{24} - x_{25} - x_{26} - x_{27} - x_{28} - x_{29} - x_{30} - \\
 & x_{31} - x_{32} - x_{33} - x_{34} - x_{35} - x_{36} + x_{59} = 0 \\
 (24) \quad & -x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - x_{43} - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - \\
 & x_{49} - x_{50} - x_{51} - x_{52} - x_{53} - x_{54} + x_{60} = 0
 \end{aligned}$$

method - converting costs to dollars per box.¹¹ This allows us to use the same coefficients as are used for variables 55-57 by using adjusted cost figures. The conversion factor, to convert \$/lb to \$/box is 1.259 lb/box. The converted figures are shown in the column labelled "GFM 2 \$/Box" in part 2 of TABLE 3.

Step 10 - Check the list of constraints for unnecessary, redundant or contradictory (inconsistent) constraints in order to reduce and simplify the model. An unnecessary constraint is one that can be excluded from the model without affecting the solution. There are no simple tests that can be applied to decide whether or not a given constraint is necessary. A rule-of-thumb is if a constraint imposes no limits on the variables of the problem or if the value of all the variables in it can be obtained from the solution to the model without that constraint the constraint is unnecessary. For example if a vendor can supply all the requirements, in general, an equation for his capacity is unnecessary. If, however, unused capacity

¹¹ Actually the cost is in dollars per amount of propellant required for one box of ammunition.

carries a cost then the constraint would be necessary. This constraint might also be included to obtain extra information about the system. Unnecessary constraints are eliminated unless retained for additional information.¹²

A formal discussion and explanation of redundancy and inconsistency is beyond the scope and intent of this paper.¹³ For our purposes it is sufficient to say that a mathematical expression is redundant if it can be obtained by multiplying another mathematical expression by a constant.¹⁴ For example the equations (1) $x_1 + 2x_2 = 5$ and (2) $2x_1 + 4x_2 = 10$ are redundant, because one can be obtained by multiplying the other by two. Another way of viewing redundancy is that all but one of the redundant constraints in a model can be eliminated without affecting the solution. That is, when equation (1) is in the model equation (2) above adds no new information and does nothing to further limit the possible solutions. Eliminate all but one of each set of redundant constraints.

For our purpose two or more mathematical expressions are inconsistent if there exists no values for the variables that satisfy the mathematical

¹² Step 5 also discusses elimination of unnecessary requirements and restrictions. The reader may also find the discussion Reinfeld or Vogel (13) on the elimination of equations and substitutions of variables enlightening.

¹³ Consult Dantzig (7) pp. 71-72 for a formal definition of redundancy and inconsistency.

¹⁴ The mathematically sophisticated reader may note that equations are redundant when they are not linearly independent.

expressions simultaneously. This really says that contradictory or mutually exclusive requirements have been placed on the variables. For example $x_1 \leq 5$ and $x_1 \geq 6$ are inconsistent. It is impossible for a variable to be simultaneously less than or equal to five and greater than or equal to six. Another example is $x_1 + x_2 = 5$ and $x_1 + x_2 = 6$ which would require a quantity of $x_1 + x_2$ to be simultaneously equal to five and six.

Inconsistent constraints tell us that the model was improperly formulated or that there is no solution to the system because the limits on the system are self contradictory. If inconsistency is encountered either the model must be reformulated or the limits of the real system must be changed. If one of these cannot be done then there is no solution to the model.

An examination of Figure 2 reveals no redundant or inconsistent mathematical expressions. Since we have already (in Step 5) eliminated those unnecessary constraints that we decided we did not want there is nothing more to do in this step.

Step 11 - Convert any inequations to equations.¹⁵ The procedure is simple. If the inequation is of the form of "less than or equal" (\leq) or "less than" ($<$) an additional variable called a slack variable is added. For those cases where the inequations are of the form " \geq " (greater than or equal to) or " $>$ " (greater than) a slack variable is subtracted and a variable called a dummy or artificial variable is added.

¹⁵The reason for doing this will be explained in the section on solutions.

In a constraint equation the values determined for the variables must be such that the sum of the terms in each constraint exactly equals the stipulated amount of that constraint. In an inequation the sum of the terms differs from the constant (b_i) value. The slack variable added or subtracted represents this (unspecified) difference.

The variables that represent the activity levels in the original equations and inequations are called real or structural variables to distinguish them from slack and artificial variables. The condition (requirement) of non-negativity imposed on real variables applies to slack and artificial variables.

The problem as formatted in Figure 2 shows all the constraints in equation form. To verify that this is indeed the case an examination of the problem and problem definition is in order. The problem, as presented for analysis, is to purchase exactly the specified number of rounds for each of the destinations. These requirements must be met exactly. Therefore the requirements for ammunition are equations. The requirements for GFM for a vendor are determined by the number of rounds of ammunition purchased from that vendor. That is, each round (or box) of ammunition requires exactly so much of each of the GFM. Therefore the GFM requirements are equations. Since the constraints are already in equation form we can proceed to Step 12.

Step 12 - Construct a coefficient table. The body of this table will be used to record the values of the coefficients of the variables. Each row in the body of the table will represent an equation, each column will represent a variable. Each equation and each variable should be labelled.

To the left of the body of the table create a column for labelling the constraints and a column for recording the value of the stipulation, the b_i , for each row. Label each row and column. Insert the values of the stipulations in the stipulations column. Record the values of the coefficients in the appropriate column within the row for that equation.

The coefficient table is shown in Figure 3. The body of the table is obtained by rewriting the coefficients of the variables without the associated variables each in the proper row and column. The stipulated amounts are from TABLE 2. The labels are from TABLE 4.

Step 13 - Determine the coefficients of the variables in the objective function and record these along the bottom of the coefficient table. The cost of a variable in the objective function is in general the sum of the costs of that term for all the rows (equations) in which it appears. This data is obtained from the table of data prepared in Step 4. It should be noted here that cost may have been given in one unit of measure and in Step 8 the unit of measure may have been changed. If this has been done it is necessary to convert the cost to the proper unit.

The bottom line of Figure 3 shows the cost coefficients. These coefficients are the cost of the variables. The cost of each of the variables $x_1 - x_{54}$ includes the cost of one box of .45 caliber ammunition and the cost of shipping it to the appropriate destination. The cost of each variable $x_{55} - x_{60}$ is the cost of shipping the appropriate GFM to the appropriate vendor. These latter costs could have been included in the costs of $x_1 - x_{54}$ and a zero cost assigned to $x_{55} - x_{60}$. However, we

wanted to identify the costs of the GFM separately. Figure 3 shows the whole model displayed in tableau form.

Step 14 - Check the model against the problem definition and the real system being simulated. Make sure that all the activities have been defined, all the constraints stated, and all the costs included.¹⁶ Ask yourself "How well does this model simulate the problem?" If it is necessary to modify the model because it is not adequate return to the proper step.

If, the reader carefully tries carrying out this step he might, if he has not already, come to the conclusion that equations 19 - 24 and variables $X_{55} - X_{60}$ could be eliminated and the costs of $X_{55} - X_{60}$ could be added to the cost of the appropriate variable for the principle items without affecting the validity of the solution. However, we retain them for the same reason we decided to back in Step 5 to use these six equations to represent the requirements of each vendor for each GFM separately instead of two equations to represent the requirements for GFM - to obtain the precise total amount of each GFM required for each vendor directly from the solution.

The total model in equation form including all the equations and the objective function after the values for all the coefficients and the stipulated amounts have been determined is shown in Figure 4.

¹⁶ Although the word all is used here it should be kept in mind that, as Ferguson and Sargeant (10) point out, in formulating a linear programming model you neglect the negligible.

Notice that except for inclusion of the objective function in Figure 4 it is exactly the same as Figure 2. This will not always be true. It is true in this case because Steps 10, 11 and 14 required no changes in the model.

The Nature of Solutions

The constraints of a linear programming problem are a system of simultaneous linear equations and linear inequations. The central mathematical problem of linear programming tells us that the solution to a linear programming problem is the solution to these simultaneous mathematical expressions which optimizes the objective function.¹⁷ Thus solving a linear programming problem requires identifying the feasible solutions - the sets of non-negative solutions which satisfy the constraints - and isolating the solution which optimizes the objective function. The first step generally is to convert the inequations to equations. This results in the constraints becoming a set of M equations in N unknowns where $N > M$.

A system of M equations in N unknowns, where $N > M$ has, in general, an infinite number of solutions. Indeed, it is this very fact that allows for the existence of alternative activities or conversely this is the result of the fact that the tasks of the system can be carried out in more than one

¹⁷

A linear programming problem may have more than one optimum solution. That is, more than one set of variables may optimize the objective function. These are referred to as alternate optimum solutions. The value of the objective function is the same (optimum) value for all the alternate optimum solutions.

way.

A system of simultaneous linear equations where the number of unknowns does not equal the number of equations or where there are a large number of either - say more than three - is most conveniently solved by the application of certain techniques and principles from matrix algebra. Specifically the properties of the inverse of a matrix¹⁸ are applied to the matrix of coefficients. This explains the tableau in Figure 3. The reader will note that this tableau is the matrix of the coefficients of the constraint equations with an additional column for the b_i (constant) values and an additional row for the cost values (cost coefficients).¹⁹ Because of the important role played by matrix algebra in solving linear programming problems the variables - represented by columns in the coefficient matrix - are often referred to as "vectors".

The problem of solving a linear programming problem reduces to the problem of finding solutions to a system of simultaneous linear equations and then systematically evaluating these solutions. From this it should be obvious that all methods of solution are essentially formalized trial and error processes. Indeed, this explains the highly iterative nature of the solution process.

¹⁸The reader unfamiliar with matrix theory is referred to Kemeny, Snell and Thompson (11) for a simple, concise, and understandable explanation of the application of matrix algebra to the problem of solving systems of linear equations.

¹⁹Now that we have introduced the idea of thinking of a linear programming model as a matrix we are in a position to explain how the dual problem is obtained. The dual problem is obtained from the direct by interchanging the rows and columns of the coefficient matrix and by interchanging the constant column with the cost row.

The Geometry of Solutions

From algebra the reader may recall that any linear expression can be represented graphically as the locus of all points that satisfy it.

For example:

$$1) 2x_1 + x_2 = 2$$

is the equation of a straight line. All those points and only those points on the given line satisfy equation (1). The strict inequation

2) $2x_1 + x_2 < 2$ is satisfied by all those points and only those points below the line representing equation (1). The inequation

3) $2x_1 + x_2 \leq 2$ is satisfied by all and only those points on and below the given line.

The set of points that satisfies a linear expression (its locus) is called a convex set or a convex region. The intersection of two or more convex sets, each of which represents a linear expression, forms a polygonal convex set or, if the area is finite, a convex polygon. A theorem from the Geometry of Convex Sets tells us that optimum - maximum or minimum - value of a linear function defined over a convex region occurs at a corner or vertex. This fact, as the reader will see, plays a crucial part in obtaining optimum solutions to linear programming problems.

The preceding discussion has been concerned with two dimensions - that is linear expressions in two unknown. Although the terminology

²⁰ Although the proof is elementary it will not be given here. If the reader is interested he can find the proof and details on the subject in the books listed in the reference. Kemeny, Snell and Thompson (11) have a particularly good - simple and concise - proof of this theorem.

must be modified to reflect the geometry of n -dimensional space the same principles apply to linear expressions in n unknowns. In other words it is true for linear expressions in more than two unknowns that their loci (or truth sets) are convex regions, the intersection of two or more convex regions is another convex region which is a convex polyhedron if the area of the intersection is finite, and a linear equation defined over a convex region has its minimum or maximum value at a vertex.

Methods of Solution

Several methods of solution are available. Some are general - that is they can be used to solve any LP problem - and others are specific - that is they can be used to solve only certain types of LP problems. In addition, some methods are exact - that is, they guarantee a best (an optimum) solution to a properly formulated linear programming problem - and some methods are approximation techniques. While differing widely in computational procedure all methods are essentially formalized trial-and-error techniques. They all involve the same general process of selecting a first solution, evaluating that solution, modifying the solution and calculating the improvement yielded from the modification. This process is continued until no further improvement can be made. The procedures for selecting the next solution generally assure that each solution is at least as good as the previous solution. A solution is modified by bringing a new vector into the solution to replace one that goes out of solution. This is sometimes referred to as "trading off". Briefly then all methods start with a feasible

answer and iteratively approach the best answer.

Leonid Hurwicz and George B. Dantzig in the summer of 1947 developed the "Simplex Method" for solving linear programming problems.²¹ The Simplex was the first method developed for solving linear programming problems. Indeed, all other methods are derived or have evolved from it. The simplex method uses matrix algebra to identify feasible solutions and a method of moving along edges of a convex polyhedron from one vertex to the next, based on an approach suggested by Fourier, to isolate the optimum solution.²²

The simplex method is awkward and cumbersome to use. It is lengthy and iterative. Consequently other methods have been developed. Two of the general and exact methods that have been developed are the Modified Simplex Method and the Dual Method. The Modified Simplex Method, also called the Inverse Matrix Method, was developed and presented by A. Charnes and C. E. Lemke (4) in 1952. As the name suggests it effects certain modifications in the simplex routine. The objectives of the modified simplex are to simplify the computational procedure by reducing the number of calculations necessary and to reduce and localize errors in calculations and round off. The Dual Method was first presented by C. E. Lemke in his doctoral dissertation in 1953. The dual method solves the problem of the dual instead

²¹The term "linear programming" was suggested to Dantzig by T. C. Koopmans to replace the longer term "programming in a linear structure" originally used by Dantzig.

²²In 1826 Fourier was faced with the problem of finding the least maximum deviation fit to a system of linear equations. He suggested finding the lowest point of the convex polyhedral set by a vertex-to-vertex descent to a minimum.

of the direct.²³

Among the exact but specialized methods of solution are the Transportation or Distribution Method, the Modified Distribution or Modi Method, and the Ratio-Analysis Method. The Transportation Method was developed to solve problems involving the distribution of a single product from several sources to several destinations. It assumes equal supply and demand and a common unit of measure. The Transportation Method is basically a computational simplification of the Simplex applied to transportation and distribution problems. The Modi Method expanded the areas of application and refined the computational procedure of the Distribution Method. The Ratio-Analysis Method provides a way of decreasing the number of computations by selecting and solving the heart or core problem of certain kinds of linear programming problems. The ratio-analysis method is a procedure for allocating limited resources among competing demands.

In addition to the exact methods of solution approximation techniques are available. Among these are the VAM (Vogel's Approximation Method) and the Index Method. These can be used either to find near optimum (suboptimum) solutions or as a method of obtaining a better starting basis to be used in one of the exact methods.

Computer programs generally use one of the general methods - especially the Simplex and the Modified Simplex Methods. The Distribution and

²³Bartlett and Charnes (1) point out that dual method applied to the dual problem coincides with the simplex method applied to the direct problem. They further point out that this is not the same as saying that the Dual Method is the simplex method applied to the dual problem.

Modi-Methods were developed at least partially to enable persons untrained in mathematics to solve LP problems by hand computations. However, for large matrices both methods are extremely tedious and time consuming. Either the VAM and the Index Method can be used in conjunction with the Modi and Distribution Methods as a means of obtaining a better starting point and thus reducing the number of iterations necessary to arrive at an optimum solution. Of course, approximation techniques are valuable in their own right where time is an important factor, frequent changes in the system occur, or the cost of an exact solution is prohibitive. Here as everywhere the law of diminishing returns may be a factor. That is, the additional savings reaped from an optimal solution over those reaped by a suboptimal solution may not justify the cost of using an exact method of solution to obtain an optimal solution.

It was stated above that some methods are exact methods. While this is true mathematically it is not necessarily true arithmetically. The reason for this is that computation almost inevitably involves decimals. In hand computations or in machine solutions this means that at some point in time rounding or truncation must be resorted to. This introduces into our exact method of solution computational errors. This fact is one of the factors that led to the development of the modified simplex. Generally, however, the errors are small in proportion to the total values involved.

Solution

The problem was originally solved on a RCA 501 using a standard computer routine. The RCA 501 linear programming routine employs the modified

simplex method. After the solution was determined it was analyzed and evaluated to assure that indeed the model was formulated so that the answer obtained was the answer to the question we wanted to ask.²⁴ The solution is shown in TABLE 6. The reader will note that all the requirements were exactly met.

In this problem it is very easy to test the solution. Look at either Figure 3 or TABLE 3 for a minute. Note that Vendor 1 submitted a higher bid for all destinations than either Vendor 2 or Vendor 3. Since any of the vendors can meet all our requirements it is obvious that Vendor 1 has priced himself out of consideration. Comparing Figure 3 (or TABLE 3) to TABLE 6 it will be noted that the vendor with the lower bid price was awarded the contract for that destination in all cases except destination 17. A glance at part 2 of TABLE 3 and a little arithmetic explains this apparent contradiction. If we add the cost of shipping the GFM to the bid prices, we find that the Vendor 2 price is \$43.287 whereas the Vendor 3 price is \$43.288. This tells that while the difference between the two prices is slight, Vendor 2 should be awarded the contract.

From the above analysis it should be obvious that the solution is indeed the least cost solution. Thus our model was properly formulated. The reader will also know why we earlier stated that the solution to this problem was trivial once it was formulated properly. Indeed, once TABLE 3

²⁴A linear programming model will always give us the correct answer to the problem we pose. However, if the model does not properly simulate the real system we are asking the wrong question and thus will get the "wrong" answer. This is why it is so important to test and evaluate the solution before implementing it.

was constructed the problem could have been solved by inspection. This is not always, indeed is very seldom, the case with LP problems.

Two things were done to the problem data for convenience of presentation that did not effect the validity of the model or the analysis thereof. In the original problem the data was presented with costs for two possible modes of transportation of the end item from vendor to destination. The lower of the two was used in our presentation and the higher ignored. Since this is a least cost problem and the capacities of the vendors and transportation facilities are not factors, this does not effect the solution.

In solving a real problem one might consider carrying both modes of transportation because at implementation it may not be possible to use the least cost source. This would make it easy to identify the next cheapest source.²⁵ When the problem was originally solved both modes of transportation were carried. However, this was not really necessary and the decision to carry both costs should consider that this would double the time to prepare the input to the computer and increase the running time to solve the problem. This cost should be weighed against possible loss of savings that would result from trial and error adjustment of solution if the optimum solution could not be implemented.

It is the general practice of procurement personnel to supply the

²⁵This is especially easy if the problem was solved on a computer. Most computer routines for solving linear programming problems include shadow prices and replacement costs so that one can easily evaluate the cost of different suboptimum solutions.

vendor with more than the required number of boxes. The rationale behind this is that some will be damaged in shipment and some may be sent that are damaged. The excess is thus an allowance for damaged boxes. The practice is to decide on the number of boxes to be allowed and prorate them to the vendors awarded contracts in direct proportion to the percentage of the total requirement of ammunition they are awarded contracts for. This was ignored in the example (but not the original model) because it did not ^affect the solution and would have required using decimal coefficients. We could have easily corrected the cost figures and GFM requirement figures in TABLE 6 by multiplying the two requirements for boxes by a correction factor.²⁶

²⁶This factor is $73,337/64,070$ where 73,337 is the total number boxes allowed and 64,070 is the number really needed - which is the figure we used.

Figure 4

- (1) $x_1 + x_{19} + x_{37} = 5,526$
- (2) $x_2 + x_{20} + x_{38} = 3,478$
- (3) $x_3 + x_{21} + x_{39} = 5,000$
- (4) $x_4 + x_{22} + x_{40} = 3,080$
- (5) $x_5 + x_{23} + x_{41} = 66$
- (6) $x_6 + x_{24} + x_{42} = 150$
- (7) $x_7 + x_{25} + x_{43} = 9,043$
- (8) $x_8 + x_{26} + x_{44} = 98$
- (9) $x_9 + x_{27} + x_{45} = 745$
- (10) $x_{10} + x_{28} + x_{46} = 14,465$
- (11) $x_{11} + x_{29} + x_{47} = 14,833$
- (12) $x_{12} + x_{30} + x_{48} = 1,000$
- (13) $x_{13} + x_{31} + x_{49} = 1,000$
- (14) $x_{14} + x_{32} + x_{50} = 1,000$
- (15) $x_{15} + x_{33} + x_{51} = 1,000$
- (16) $x_{16} + x_{34} + x_{52} = 1,000$
- (17) $x_{17} + x_{35} + x_{53} = 1,000$
- (18) $x_{18} + x_{36} + x_{54} = 1,586$
- (19) $-x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - x_{14} -$
 $x_{15} - x_{16} - x_{17} - x_{18} + x_{55} = 0$
- (20) $-x_{19} - x_{20} - x_{21} - x_{22} - x_{23} - x_{24} - x_{25} - x_{26} - x_{27} - x_{28} - x_{29} - x_{30} -$
 $x_{31} - x_{32} - x_{33} - x_{34} - x_{35} - x_{36} + x_{56} = 0$
- (21) $-x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - x_{43} - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} -$
 $x_{49} - x_{50} - x_{51} - x_{52} - x_{53} - x_{54} + x_{57} = 0$

$$(22) -x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - x_{14} - x_{15} - x_{16} - x_{17} - x_{18} + 58 = 0$$

$$(23) -x_{19} - x_{20} - x_{21} - x_{22} - x_{23} - x_{24} - x_{25} - x_{26} - x_{27} - x_{28} - x_{29} - x_{30} - x_{31} - x_{32} - x_{33} - x_{34} - x_{35} - x_{36} + x_{59} = 0$$

$$(24) -x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - x_{43} - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - x_{49} - x_{50} - x_{51} - x_{52} - x_{53} - x_{54} + x_{60} = 0$$

Objective Function

$$\begin{aligned} &46.68x_1 + 47.69x_2 + 67.08x_3 + 46.79x_4 + 50.10x_5 + 48.68x_6 + 47.36x_7 + 48.75x_8 + \\ &46.99x_9 + 46.99x_{10} + 47.36x_{11} + 46.93x_{12} + 46.79x_{13} + 46.94x_{14} + 66.86x_{15} + \\ &66.88x_{16} + 46.48x_{17} + 47.29x_{18} + 43.03x_{19} + 44.29x_{20} + 43.75x_{21} + 43.49x_{22} + \\ &45.94x_{23} + 45.03x_{24} + 44.25x_{25} + 45.09x_{26} + 43.83x_{27} + 43.56x_{28} + 44.25x_{29} + \\ &43.43x_{30} + 43.46x_{31} + 43.50x_{32} + 43.57x_{33} + 43.49x_{34} + 43.18x_{35} + 44.25x_{36} + \\ &43.08x_{37} + 44.60x_{38} + 43.26x_{39} + 43.66x_{40} + 47.60x_{41} + 43.89x_{42} + 44.40x_{43} + \\ &44.83x_{44} + 42.64x_{45} + 42.59x_{46} + 44.40x_{47} + 43.21x_{48} + 43.04x_{49} + 42.70x_{50} + \\ &43.82x_{51} + 43.61x_{52} + 43.17x_{53} + 44.40x_{54} + 0.131x_{55} + 0.087x_{56} + 0.099x_{57} + \\ &0.025x_{58} + 0.020x_{59} + 0.019x_{60} = z \end{aligned}$$

TABLE 6

Solution to .45 Caliber Problem

<u>Destination Points</u>	<u>Vendor 2</u>		<u>Vendor 3</u>	
	<u>Units</u>	<u>Cost</u>	<u>Units</u>	<u>Cost</u>
1	5,526	\$237,783.78		
2	3,478	154,040.62		
3			5,000	\$216,300.00
4	3,080	133,949.20		
5	66	3,032.04		
6			150	6,583.50
7	9,043	400,152.75		
8			98	4,295.34
9			745	31,766.80
10			14,465	616,064.35
11	14,833	656,360.25		
12			1,000	43,210.00
13			1,000	43,040.00
14			1,000	42,700.00
15	1,000	43,470.00		
16	1,000	43,490.00		
17	1,000	43,180.00		
18	<u>1,586</u>	<u>70,180.50</u>		
	40,612	\$1,785,739.14	23,458	\$1,003,959.99

TABLE 6 (Cont')

Vendor 2

GFM Items

#1 - Boxes	40,612 boxes	3,533.24
#2 - Propellant	51,131.7 lbs.	<u>818.11</u>
	Total Cost GFM	4,351.35

Total Cost \$1,790,090.49

Vendor 3

GFM Items

#1 - Boxes	23,458 boxes	2,322.34
#2 - Propellant	29,534.3 lbs.	<u>443.01</u>
	Total Cost GFM	2,765.35

Total Cost \$1,006,725.34

GRAND TOTAL \$2,796,815.83

Summary

The reader has been introduced to Operations Research, its purpose and some of its methods. One of these methods, linear programming, has been discussed in some detail. The underlying concept and the mathematical nature of linear programming have been detailed. While the author has not always been mathematically rigorous every effort has been expended to insure the mathematical precision (validity) of the material presented.

In addition a detailed method of model building was presented. Following this method of model building results in the problem being stated in

its matrix form. Some methods of solution - especially the specific methods - require that the problem be formatted differently. Inequations are handled somewhat differently in some methods. Most computers program use a general method and require the problem be stated in its matrix form. Even if the method used requires formatting the problem differently we recommend first building a tableau as per the method of this paper because it is a relatively simple task to transform the model to the required form and the tableau offers an excellent visual aid to help the builder see the interrelations between the variables. Sometimes, in fact, the problem can be solved by inspection once it is formulated into a matrix.

PART II

Nonlinear Programming

CHAPTER 3

Introduction

Linear programming is a very useful technique that can be used to solve (least cost) procurement problems. Unfortunately, however, not all procurement problems are linear. The present chapter will be devoted to presenting a solution to a nonlinear programming problem.

A numerical analysis approach will be used. While the method presented will not be completely general the general principles underlying the approach will be pointed out.

The Problem

For this problem we have a single total requirement for a certain number of rounds of 7.62 mm (NATO-round) ammunition. While all of the vendors collectively can supply the total requirement none of them can supply it individually. There are five (5) vendors that submitted bids, two (2) submitted bids at more than one level. The objective is to purchase exactly the required number of rounds at the least cost. The objective is influenced by the following factors:

- 1) the unit price of the end item.
- 2) the charge for the use of government owned property that is used by the vendor in the manufacturing of the end item.
- 3) the cost of shipping GFM to the vendors.

NOTE: The cost of shipping the packaged end item is not an influencing factor in this model.

An unusual characteristic of this model is the cost factor for each of the vendors. It is expressed as $a_i + b_i^j x_i$. The a_i represents the use charge (the charge for the use of Government owned property) which is incurred when the i th vendor is brought into solution. The b_i^j represents the combined unit price of the end item which is the unit cost of shipping GFM to the i th vendor and the bid price for the j th level (if a multilevel bidder). Therefore, the equation $a_i + b_i^j x_i$ states that when a vendor is brought into solution for whatever amount " x_i " (a quantity of the end item for the i th vendor) there is a constant cost a_i (which is different per each vendor) plus a variable cost $b_i^j x_i$.

Background

The use charges are fixed costs associated with a given vendor. A use charge is incurred as soon as one unit is purchased from a given vendor and the amount is not dependent on the number of rounds purchased from him. These are easily handled.

The multilevel bidding presents a different problem. Multilevel bids are those bids that quote different prices for different quantities of end item. The term bidder level will be used to refer to either a particular level of a multilevel bid or to the bid of single level bidder. Two types of multilevel bidding are possible. These are the graduated multilevel bid and the discrete multilevel bid. In the case of discrete multilevel bids the price changes for the total quantity purchased as one goes from one quantity level to the next. In graduated bids the different price applies to only the increment of quantity when one goes from one quantity level to the next.

The NATO-round problem involves discrete bids. This fact, together with the fact that one of the bidders submitted a minimum quantity that was different from zero made the 7.62 round problem nonlinear.

When the problem was first presented for analysis it was set up in standard linear programming format. The nonlinear characteristic of the problem became obvious when it was attempted to solve the problem by LP. The authors then attempted to express the problem in linear form. Lack of time forced abandonment of the attempts to simulate the system with a linear model. Since no general method for solving nonlinear programming problems exist a review of the technical literature was undertaken in an attempt to find a method of solution.

Bellman's Dynamic Programming algorithm could be modified to solve the problem. However, it would not guarantee an optimum solution and would require a prohibitive amount of time to solve the problem. The only other method suggested was to break the model up into a series of models each model containing one and only one level from each bidder. This would have required setting up and solving eight models.²⁷ It was decided therefore to use a method of numerical analysis.

Method and Underlying Assumptions

The method discussed is a numerical analysis method. The general process involved is similar to that of solving a linear programming problem. First the problem is defined, then the data is formatted, a first solution

²⁷The optimum of the optimum solutions of the smaller models would be an optimum of the larger model. See Vadja (14) for details.

is computed, a modification is made of the solution, and the effect of the modification is evaluated. This is continued until no further improvement is possible.

Two assumptions regarding the handling of the costs have been made. The quantity of GFM needed by a vendor is dependent on the quantity of end item allocated to him. The first assumption, therefore, is that the cost of shipping GFM can be put on a per unit basis. The costs of shipping GFM were originally given as total costs for the maximum number of rounds for each bidder. To put these costs on a per unit basis divide the total cost (for that bidder) by the maximum limit of that bidder. Secondly, it is assumed that use charge can be prorated over the maximum quantity for each bidder level.

The problem data is presented in TABLE 7. Note that the GFM cost was prorated over the total (maximum) quantity whereas the use charge was prorated over the maximum quantity for each level. The use charge is a flat rate that applies regardless of number of units bought whereas the GFM (shipping) cost is a per unit charge. In other words the use charge per unit is inversely proportional to number of units purchased and the GFM cost per unit is constant. The total use charge is constant and the total GFM cost is directly proportional to the number of units purchased. Prorating the use charge will sometimes (when less than the maximum quantity for a bidder level is involved) introduce errors. However, in our problem (and in general) the use charge is very small in comparison to the total cost.²⁸ Therefore, this error will not hinder the process of selecting

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Note that in this problem that total use charge is in the magnitude of \$145 thousand dollars whereas the total minimum cost will be greater than \$14.6 million.

TABLE 7

Vendor	Unit Price	Quantity range in millions of rds	PROBLEM DATA			Cost of shipping GFM to vendor (\$ for max. quant) (\$/rd)
			Quantity range in millions of rds	Charge for the use of Government Owned Property (total charge)	Charge for the use of Property (\$/rd)	
A	\$0.06101	0 - 33		\$ 3,855.84	\$0.000117	\$ 4,620.28 \$0.000114
B	0.06189	0 - 165.6		13,456.00	0.000081	21,350.51 0.00013
C	0.06199	150 - 160			0.000786	
	0.06396	100 - 150		125,804.84	0.000839	20,630.47 0.00013
	0.06592	70 - 100			0.001258	
	0.06797	22 - 70			0.001797	
D	0.07050	0 - 12		6,583.98	0.000549	2,938.94 0.00024
E	0.06800	42 - 77		NONE		20,919.69 0.00027
	0.07000	0 - 42				

Required Quantity: 239,600,000 rds.

the proper bidder level. The cost figure will be corrected after each iteration to reflect the fact that the total use charge may not be included in it.

In addition, two assumptions regarding the procedure to be used were made. First, the various levels of a multilevel bidder can be treated as individual bidders with the proviso that no more than one level from any given vendor can be in the solution at one time. The second assumption is that the bidder levels can be ranked on the basis of total unit cost and a first solution derived by assigning the maximum possible quantity to each bidder level beginning with the least expensive bidder level.

It should be pointed out the total unit cost is used only for the purpose of ranking the bidder levels. In evaluating a solution or determining gain or loss resulting from a modification the formula $a_i + b_i^j$ is used where a_i is the total use charge for the i th bidder and b_i^j is the sum of the per unit price of the end item and the per unit cost of shipping the GFM (column labelled $a + b$ in Figure 5).

The Solution

The first step is to determine the appropriate per unit costs for use charges and shipping GFM. These figures are shown in TABLE 7. The next step is to combine the three costs - the unit price of the end item and the per unit cost of shipping GFM, and the per unit use charge - into a single total unit cost or value. The various bidder levels are then ranked on the basis of the total unit cost. Figure 5 shows the bidder levels arranged properly. The quantity range covered by each

bidder level, unit price, unit cost of shipping GPM, and total unit price are all shown.

Now the problem is set up and we are ready to begin. The fourth step is to assign to each bidder level beginning with the bidder level with the lowest total unit price the maximum quantity for that level. The total requirement is for 239.6×10^6 rounds. We assign 33.0×10^6 to Vendor A and 165.6×10^6 to Vendor B. This accounts for 198.6×10^6 rounds. Note how this assignment also agrees with our intuition. This is 41.0×10^6 rounds below the requirement. The first three levels of Vendor C and the first level of Vendor E cannot be considered because their minimum quantity levels are greater than our unfulfilled requirement. The fourth level of Vendor C is next lowest priced. This level has a minimum less than and a maximum greater than our requirement. Therefore, we allocate the entire 41.0×10^6 rounds to the fourth level of Vendor C. This completes our first solution. This solution is shown in Figure 5 - together with its total cost - under column labelled Trial 1.

The existence of a better - lower cost - solution is based on the fact that it may be worthwhile to reduce or eliminate Vendor A or Vendor B in order that C-4 can be replaced by E-1, C-3, C-2 or C-1. We shall successively bring these into solution and evaluate the effect of the modification on the objective function.

The cheapest way to eliminate C-4 and bring E-1 into the solution is to reduce the number assigned to Vendor B. This has been done and the result evaluate in trial 2 of Figure 5. Note that "vendor" E-1

Figure 5

SOLUTION TO NATO ROUND PROBLEM

Vendor & Level	Quantity in million rds	Unit Costs \$/rd				Total (a+b+c)	Use Cost \$/Vendor	Trial					#6
		Price (a)	GM (b)	Use (c)	(a+b)			#1	#2	#3	#4	#5	
A	0 - 33	.06101	.00014	.00012	.06115	.06129	\$ 3,855.84	33.0	33.0	33.0	33.0	33.0	
B	0 - 165.6	.06189	.00013	.00008	.06202	.06210	13,456.00	165.6	164.6	136.6	106.6	56.6	89.6
C-1	150 - 160	.06199	.00013	.00079	.06212	.06291	125,804.84					150.0	150.0
C-2	100 - 150	.06396	.00013	.00084	.06409	.06493	125,804.84				100.0		
C-3	70 - 100	.06592	.00013	.00126	.06605	.06731	125,804.84			70.0			
E-1	42 - 77	.06800	.00027	0	.06827	.06827			42.0				
C-4	22 - 70	.06797	.00013	.00180	.06810	.06990	125,804.84	41.0					
E-2	0 - 42	.07000	.00027	0	.07027	.07027							
D	0 - 12	.07050	.00024	.00055	.07074	.07129	6,853.98						
Total Quantity								239.6	239.6	239.6	239.6	239.6	239.6
Evaluated Cost								\$15,223,678.68	\$15,111,093.84	\$15,256,498.68	\$15,181,398.68	\$14,988,398.68	\$15,018,107.84

was brought into solution for its minimum quantity. To bring it into solution for any more than the minimum would - in this case - be a less optimum solution because it would mean replacing Vendor A or Vendor B with "vendor" E-1 on a one for one basis at a higher unit cost.

The trial and error process is continued with "vendor" C-3 replacing "vendor" E-1 in trial 3, "vendor" C-2 replacing "vendor" C-3 in trial 4, and "vendor" C-1 replacing C-2 in trial 5. The resulting total evaluated costs are shown along the bottom of Figure 5. The results of the modifications could be evaluated by comparing total costs or by calculating gain or loss. For example, trial 3 is not as good as trial 2 because it requires buying 42.0×10^6 rounds at a savings of \$0.00222 per round and buying 28.0×10^6 rounds at a cost of \$0.00403 per round plus the additional \$125,804.84 set up cost incurred.

The solution in trial 5 is to purchase 33.0×10^6 from Vendor A, 56.6×10^6 from Vendor B, and 150.0×10^6 from Vendor C-1. This is an optimum solution. To see that this is an optimum solution let us examine the solution and analyze the possibility of further changes. We have already shown in trials 1 - 4 that increased amounts purchased from Vendor B instead of from Vendor C result in less optimal solutions. It remains only to evaluate the possibility of eliminating Vendor A and purchasing the rounds from Vendor B - thus saving the \$3,855.84 use charge associated with Vendor A. A look at Figure 5 assures us that this will result in a less optimal solution because the total cost of a round from Vendor B

before the use charge is included is greater than the cost of a round from Vendor A with the use charge included. Therefore, the solution to trial 5 is an optimum solution. Trial 6 shows the results of eliminating Vendor A entirely from the solution. Indeed, a purchase of any number of rounds from Vendor B instead of Vendor A will adversely affect the solution. Note that if the unit cost of Vendor B is less than Vendor A (when use charges are not included in unit cost) then eliminating Vendor A would indeed provide a better solution. This is why it is very important to evaluate each iteration on the basis of actual costs.

Summary and Remarks

We have presented a method of solution to certain kinds of nonlinear problems. A similar approach can be used to solve other types of nonlinear programming problems. The approach was based on the assumption that the costs of shipping GPM and the use charge can be prorated, that once these are prorated on a per unit basis a total per unit cost can be calculated, and that a first solution can be determined by ranking the bidder levels on the basis of total unit cost and allocating the maximum possible quantity to each bidder level beginning with the bidder level with the least cost per unit. An optimum solution can then be determined by successive trial and error. While the method presented is not completely general and has not been rigorously proved the authors have tried to make the general principles involved sufficiently obvious so that the reader can apply the method to a large variety of problems.

One word of caution should be interjected regarding use charges. The

total cost for use of government owned property is generally small in comparison to the total cost of the requirement and the per unit cost for use of government owned property is generally small when compared to the total per unit cost. This may not always be true. However, our method of evaluating solutions on the basis of actual costs was designed to preclude the possibility that the use charge will hinder the determining of the true optimum solution.

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